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Factorization systems as Eilenberg–Moore algebras

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Abstract

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A functorial notion of factorization system is introduced and shown to coincide with the appropriate 2-categorical notion of algebra, with respect to the monad on **Cat** which assigns to a category $\mathcal{X}$ its morphism category $\mathcal{X}^2$. Quite surprisingly, it is also equivalent to the common notion of $(\mathcal{E}, \mathcal{M})$-factorization system with the usual diagonalization property; in brief: functorial choice of 'diagonals' necessarily yields their uniqueness.

Introduction

The notion of an $(\mathcal{E}, \mathcal{M})$*-factorization system* or *orthogonal factorization system* for morphisms is well established in Category Theory. Although its roots are already present in Isbell's work of the fifties (cf. [5]), systematic accounts of essential properties appeared only much later, for example in [1, 3, 6–8]. For classes $\mathcal{E}$ and $\mathcal{M}$ of morphisms (which contain all isomorphisms and are closed under composition with isomorphisms), such a system gives, for every morphism f, a decomposition $f = m_f e_f$ with $e_f \in \mathcal{E}$ and $m_f \in \mathcal{M}$ such that the diagonalization property holds: every commutative diagram

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$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, "e"'] & \bullet \arrow[d, "m"] \\ \bullet \arrow[r, "v"] & \bullet \end{tikzcd}$$

with $e \in \mathscr{E}$ and $m \in \mathscr{M}$ admits a uniquely determined fill-in morphism t with $te = u$ and $mt = v$. The system is completely determined by each $\mathscr{E}$ and $\mathscr{M}$, and both classes enjoy all good stability properties which one expects them to have:

- $\mathscr{M}$ is closed under limits, and $\mathscr{E}$ is closed under colimits,
- $\mathscr{E}$ and $\mathscr{M}$ are closed under composition.

As was pointed out by Im and Kelly [4], closedness under limits implies a number of other important stability properties (for instance, stability under pullback and weak cancellation).

For the study of weak notions of factorization system, however, it seems necessary to change the view on which should be the primary data determining the system. As in [2], we may start with an assignment

$$f \mapsto (e_f, m_f)$$

rather than with a class $\mathscr{E}$ or $\mathscr{M}$, taking the view that the decomposition of f itself is more fundamental than the question to which classes the factors *may* belong. As a minimal requirement then, we assume that this assignment yields for an *iso*morphism f of the category $\mathscr{X}$ a pair of isomorphisms of $\mathscr{X}$, and that it is functorial when the morphism f is considered an object of the category $\mathscr{X}^2$ of morphisms of $\mathscr{X}$, with $2 = \{\cdot \to \cdot\}$. Hence each commutative diagram

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "v"] & \bullet \end{tikzcd} \tag{1}$$

should give a commutative diagram

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, "e_f"'] & \bullet \arrow[d, "e_g"] \\ \bullet \arrow[r, "t"] \arrow[d, "m_f"'] & \bullet \arrow[d, "m_g"] \\ \bullet \arrow[r, "v"] & \bullet \end{tikzcd} \tag{2}$$

with t functorially depending on u and v. We call such a system a *weak factorization system* of $\mathscr{X}$. Putting

$$\mathscr{E} = \{h \mid m_h \text{ iso}\} \quad \text{and} \quad \mathscr{M} = \{h \mid e_h \text{ iso}\},$$

one may then ask whether each of the following properties holds:

(I) e_f belongs to $\mathscr{E}$ (for every f),
(II) m_f belongs to $\mathscr{M}$ (for every f),
(III) t is uniquely determined (by f, g, u, v and by the commutativity conditions of (2)).

It is easily checked (and well known) that the weak factorization systems satisfying conditions (I), (II), (III) provide an equivalent description of the orthogonal factorization systems mentioned before.

The question arises whether conditions (I), (II), (III) are logically independent. It is easily seen that (I) & (III) does not imply (II) in general (in a category in which regular epimorphisms are not closed under composition, such as **Cat**, factor each morphism through the coequalizer of its kernel pair). Dually, (II) & (III) does not imply (I) in general. What about (I) & (II)$\Rightarrow$(III) then? Quite surprisingly, we show that the implication does hold true in general:

Theorem A. *The orthogonal factorization systems are equivalently described by the weak factorization systems satisfying* (I) *and* (II).

In other words: *functorial choice of 'diagonals' already implies unique choice.* Finally we give a third (non-trivial) characterization of orthogonal factorization systems which provides an abstract and purely 2-categorical description of these systems:

Theorem B. *The orthogonal factorization systems are equivalently described by the* (*appropriately defined*) *Eilenberg–Moore algebras with respect to the monad which belongs to the endofunctor* $\mathscr{X} \mapsto \mathscr{X}^2$ *of* (*the* 2-*category*) **Cat**.

There are a few by-products of this result. For example, since every algebra is a quotient of a free algebra, there is a fundamental role which the *free factorization system* on $\mathscr{X}^2$ plays in these discussions; it gives the decomposition

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "v"'] & \bullet \end{tikzcd} \quad = \quad \begin{tikzcd} \bullet \arrow[r, "1"] \arrow[d, "f"'] & \bullet \arrow[r, "u"] \arrow[d, "d"] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "v"'] & \bullet \arrow[r, "1"'] & \bullet \end{tikzcd} \tag{3}$$

with $d = gu = vf$. In particular, for every f, it yields the *generic* decomposition

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "1"'] & \bullet \arrow[d, "1"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd} \quad = \quad \begin{tikzcd} \bullet \arrow[r, "1"] \arrow[d, "1"'] & \bullet \arrow[r, "f"] \arrow[d, "f"] & \bullet \arrow[d, "1"] \\ \bullet \arrow[r, "f"'] & \bullet \arrow[r, "1"'] & \bullet \end{tikzcd} \tag{4}$$

For an arbitrary system, the decomposition $f = m_f e_f$ is obtained by application of its defining functor $F : \mathscr{X}^2 \to \mathscr{X}$ to the generic decomposition. In particular, the assignment $f \mapsto (e_f, m_f)$ is already determined by the assignment

$$f \mapsto \text{codomain}(e_f) = \text{domain}(m_f)$$

in this functorial setting.

We give a fairly systematic account of the 2-categorical background material which is useful in deriving the two theorems. Readers interested only in Theorem A should be able to understand its proof by reading just Section 3, using Sections 1 and 2 only for reference of notation. However, the study of the techniques provided in Sections 1 and 2 leads to a deeper understanding of both theorems and makes the proofs more transparent.

1. The comonoid 2 and its induced monad on Cat

The category $\mathcal{K}^2$ with $2 = \{0 \to 1\}$ and $\mathcal{K}$ an arbitrary category has as objects all morphisms of $\mathcal{K}$, and a morphism $(u, v) : f \to g$ in $\mathcal{K}^2$ is given by the commutative square (1). Properties of the passage from $\mathcal{K}$ to $\mathcal{K}^2$ arise from simple facts on the category 2, which is a complete (and cocomplete) ordered set.

1.1. The functor $e : 2 \to 1$ has both a left and a right adjoint $d_0 \dashv e \dashv d_1$, with

$$ed_0 = 1 = ed_1 \,, \tag{5}$$

and with counit $\bar{\eta} : d_0 e \to 1$ and unit $\bar{\mu} : 1 \to d_1 e$, satisfying the triangular equations

$$\bar{\eta} d_0 = 1 \,, \quad e\bar{\eta} = 1 \,, \quad \bar{\mu} d_1 = 1 \,, \quad e\bar{\mu} = 1 \,. \tag{6}$$

The unique transformation $\bar{\kappa} : d_0 \to d_1$ satisfies the identities

$$\bar{\eta} d_1 = \bar{\kappa} = \bar{\mu} d_0 \,, \quad e\bar{\kappa} = 1 \,, \quad \bar{\kappa} e = \bar{\mu}\bar{\eta} \,. \tag{7}$$

1.2. Like every object of **Cat**, 2 has the structure of a comonoid $(2, e, m)$, with the comultiplication $m : 2 \to 2 \times 2$ given by the diagonal. Hence one has

$$(e, 1)m = 1 = (1, e)m \quad \text{and} \quad (m \times 1)m = (1 \times m)m \,; \tag{8}$$

here $(e, 1),(1, e) : 2 \times 2 \to 2$ are the projections.

Again, m has both adjoints $l \dashv m \dashv r$, with

$$lm = 1 = rm \,, \tag{9}$$

and with counit $\bar{\psi} : mr \to 1$ and unit $\bar{\varphi} : 1 \to ml$ such that

$$l\bar{\varphi} = 1 , \quad \bar{\varphi}m = 1 , \quad r\bar{\psi} = 1 , \quad \bar{\psi}m = 1 . \tag{10}$$

The functors $l, r : 2 \times 2 \to 2$ are given explicitly by

$$l(i, j) = i \vee j , \qquad r(i, j) = i \wedge j . \tag{11}$$

There is a unique transformation $\bar{\tau} : r \to l$, which also satisfies the identities

$$r\bar{\varphi} = \bar{\tau} = l\bar{\psi} , \quad \bar{\tau}m = 1 , \quad m\bar{\tau} = \bar{\varphi}\bar{\psi} . \tag{12}$$

1.3. The internal-hom of the cartesian closed category **Cat** is a 2-functor $\mathbf{Cat}^{\mathrm{op}} \times \mathbf{Cat} \to \mathbf{Cat}$ and gives therefore rise to a 2-functor

$$\Psi : \mathbf{Cat}^{\mathrm{op}} \to [\mathbf{Cat}, \mathbf{Cat}]$$

which sends A to $(\ )^A$; here the codomain of Ψ is the 2-category of endo-2-functors, 2-natural transformations, and modifications. Ψ is *strong monoidal*, since

$$\Psi(A \times B) = (\ )^{A \times B} \cong ((\ )^B)^A = \Psi(A)\Psi(B) .$$

Next we shall apply Ψ to the data of 1.1 and 1.2, hence only the case $A = B = 2$ will be of interest to us.

1.4. The 2-functor $\Phi = (\ )^2 : \mathbf{Cat} \to \mathbf{Cat}$ belongs to a 2-monad $(\Phi, E, M) = (\Psi(2), \Psi(e), \Psi(m))$, since (8) translates into

$$M(E\Phi) = 1 = M(\Phi E) \quad \text{and} \quad M(M\Phi) = M(\Phi M) . \tag{13}$$

Both 2-natural transformations $E : 1 \to \Phi$ and $M : \Phi\Phi \to \Phi$ have both adjoints,

$$\partial_1 \dashv E \dashv \partial_0 , \qquad R \dashv M \dashv L \tag{14}$$

with $\partial_i = \Psi(d_i)$, $L = \Psi(l)$, $R = \Psi(r)$. From (5) and (9) one has

$$\partial_0 E = 1 = \partial_1 E , \qquad MR = 1 = ML . \tag{15}$$

Writing η for $\Psi(\bar{\eta})$, μ for $\Psi(\bar{\mu})$, and so on, we have modifications $E\partial_0 \xrightarrow{\eta} 1 \xrightarrow{\mu} E\partial_1$ and $\partial_0 \xrightarrow{\kappa} \partial_1$, and the identities (6) and (7) translate into

$$\begin{aligned} &\partial_0 \eta = 1 , \quad \eta E = 1 , \quad \partial_1 \mu = 1 , \quad \mu E = 1 , \\ &\partial_1 \eta = \kappa = \partial_0 \mu , \quad \kappa E = 1 , \quad E\kappa = \mu\eta . \end{aligned} \tag{16}$$

Furthermore, for

$$RM \xrightarrow{\psi} 1 \xrightarrow{\varphi} LM \quad \text{and} \quad R \xrightarrow{\tau} L \,, \tag{17}$$

one obtains from (10) and (12)

$$\begin{aligned} &\varphi L = 1 \,, \quad M\varphi = 1 \,, \quad \psi R = 1 \,, \quad M\psi = 1 \,, \\ &\varphi R = \tau = \psi L \,, \quad M\tau = 1 \,, \quad \tau M = \varphi\psi \,. \end{aligned} \tag{18}$$

1.5. Evaluation of the data of 1.4 at a category $\mathcal{K}$ gives functors $E = E_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}^2$ and $M = M_{\mathcal{K}} : (\mathcal{K}^2)^2 \to \mathcal{K}^2$, and (13) now reads as

$$M_{\mathcal{K}} E_{\mathcal{K}^2} = \mathrm{Id}_{\mathcal{K}^2} = M_{\mathcal{K}}(E_{\mathcal{K}})^2 \quad \text{and} \quad M_{\mathcal{K}} M_{\mathcal{K}^2} = M_{\mathcal{K}}(M_{\mathcal{K}})^2 \,. \tag{19}$$

$E_{\mathcal{K}}$ and $M_{\mathcal{K}}$ have adjoints as in (14), satisfying the identities (15). Likewise, there are (ordinary) natural transformations $\eta = \eta^{\mathcal{K}}$, $\mu = \mu^{\mathcal{K}}$ and so on, satisfying the identities (16) and (18).

1.6. The embedding $E = E_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}^2$ takes $\cdot \xrightarrow{f} \cdot$ to

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "1"'] & \bullet \arrow[d, "1"] \\ \bullet \arrow[r, "f"] & \bullet \end{tikzcd} \tag{20}$$

$\partial_0 : \mathcal{K}^2 \to \mathcal{K}$ and $\partial_1 : \mathcal{K}^2 \to \mathcal{K}$ are the domain and codomain functors which assign to the $\mathcal{K}^2$-morphism (1) the $\mathcal{K}$-morphisms u and v, respectively. The natural transformation $\kappa : \partial_0 \to \partial_1$ satisfies $\kappa_f = f$ for all morphisms f in $\mathcal{K}$. The natural transformations $\eta = \eta^{\mathcal{K}} : E\partial_0 \to \mathrm{Id}_{\mathcal{K}^2}$ and $\mu = \mu^{\mathcal{K}} : \mathrm{Id}_{\mathcal{K}^2} \to E\partial_1$ are given by

$$\eta_f = (1, f) : 1 \to f \,, \quad \text{i.e.} \quad \begin{tikzcd} \bullet \arrow[r, "1"] \arrow[d, "1"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "f"] & \bullet \end{tikzcd} \tag{21}$$

$$\mu_f = (f, 1) : f \to 1 \,, \quad \text{i.e.} \quad \begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "f"'] & \bullet \arrow[d, "1"] \\ \bullet \arrow[r, "1"] & \bullet \end{tikzcd} \tag{22}$$

1.7. Let us now describe explicitly the category $(\mathcal{K}^2)^2$. Its objects are the $\mathcal{K}^2$-morphisms $(u, v) : f \to g$, and a $(\mathcal{K}^2)^2$-morphism $((a, b),(c, d)) : (u, v) \to (u', v')$ with $(u', v') : f' \to g'$ in $\mathcal{K}^2$ is given by a commutative diagram

$$
\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[d, "f"'] \arrow[ddrr, "a"] & \bullet \arrow[d, "g"] \arrow[ddrr, "c"] & & \\
\bullet \arrow[r, "v"] \arrow[ddrr, "b"'] & \bullet \arrow[ddrr, "d"] & & \\
& & \bullet \arrow[r, "u'"] \arrow[d, "f'"'] & \bullet \arrow[d, "g'"] \\
& & \bullet \arrow[r, "v'"'] & \bullet
\end{tikzcd}
\tag{23}
$$

in $\mathscr{K}$. The functor $M = M_{\mathscr{K}} : (\mathscr{K}^2)^2 \to \mathscr{K}^2$ is given on objects by the assignment

$$((u, v) : f \to g) \mapsto gu = vf \,, \tag{24}$$

and it assigns to the morphism (23) the $\mathscr{K}^2$-morphism $(a, d) : vf \to v'f'$.

Furthermore, using (11), we see that the embeddings $L = L_{\mathscr{K}}, R = R_{\mathscr{K}} : \mathscr{K}^2 \to (\mathscr{K}^2)^2$ are given explicitly by

$$Lf = \mu_f \quad \text{and} \quad Rf = \eta_f \,. \tag{25}$$

We note that there are also the following embeddings of $\mathscr{K}^2$ into $(\mathscr{K}^2)^2$:

$$
E_{\mathscr{K}^2} : \mathscr{K}^2 \to (\mathscr{K}^2)^2 \,, \qquad f \mapsto 1_f =
\begin{tikzcd}
\bullet \arrow[r, "1"] \arrow[d, "f"'] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "1"] & \bullet
\end{tikzcd}
\tag{26}
$$

$$
(E_{\mathscr{K}})^2 : \mathscr{K}^2 \to (\mathscr{K}^2)^2 \,, \qquad f \mapsto E_{\mathscr{K}} \kappa_f =
\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "1"'] & \bullet \arrow[d, "1"] \\
\bullet \arrow[r, "f"] & \bullet
\end{tikzcd}
\tag{27}
$$

For future reference we note the identities

$$(E_{\mathscr{K}})^2 R_{\mathscr{K}} = R_{\mathscr{K}^2}(E_{\mathscr{K}})^2 \quad \text{and} \quad (M_{\mathscr{K}})^2 R_{\mathscr{K}^2} = R_{\mathscr{K}} M_{\mathscr{K}} \,; \tag{28}$$

of course, similar identities hold for L.

The $\mathscr{K}^2$-morphism $\tau_f : Rf \to Lf$ is represented by the diagram

$$
\begin{tikzcd}
\bullet \arrow[r, "1"] \arrow[d, "1"'] \arrow[ddrr, "1"] & \bullet \arrow[d, "f"] \arrow[ddrr, "1"] & & \\
\bullet \arrow[r, "f"] \arrow[ddrr, "f"'] & \bullet \arrow[ddrr, "1"] & & \\
& & \bullet \arrow[r, "f"] \arrow[d, "f"'] & \bullet \arrow[d, "1"] \\
& & \bullet \arrow[r, "1"'] & \bullet
\end{tikzcd}
\tag{29}
$$

Finally, the natural transformation φ and ψ of (17) are described explicitly by

$$\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[d, "f"'] \arrow[ddrr, "1"] & \bullet \arrow[d, "g"] \arrow[ddrr, "g"] & & \\
\bullet \arrow[r, "v"] \arrow[ddrr, "v"'] & \bullet \arrow[ddrr, "1"] & & \\
& & \bullet \arrow[r, "vf"] \arrow[d, "vf"'] & \bullet \arrow[d, "1"] \\
& & \bullet \arrow[r, "1"] & \bullet
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r, "1"] \arrow[d, "1"'] \arrow[ddrr, "1"] & \bullet \arrow[d, "vf"] \arrow[ddrr, "u"] & & \\
\bullet \arrow[r, "vf"] \arrow[ddrr, "f"'] & \bullet \arrow[ddrr, "1"] & & \\
& & \bullet \arrow[r, "u"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\
& & \bullet \arrow[r, "v"] & \bullet
\end{tikzcd}$$

(30)

2. Weak factorization systems

2.1. We start off with the weakest functorial notion of factorization system that seems reasonable and call a functor $F : \mathcal{K}^2 \to \mathcal{K}$ a *weak factorization system* of $\mathcal{K}$ if there is a natural isomorphism $\gamma : \mathrm{Id}_{\mathcal{K}} \to FE_{\mathcal{K}}$. Hence, when putting $e_f = F\eta_f \cdot \gamma_A$ and $m_f = \gamma_B^{-1} \cdot F\mu_f$ for $f : A \to B$ in $\mathcal{K}$, from the last identity of (16) and the naturality of γ we get $m_f \cdot e_f = f$. The naturality of $F\eta$, $F\mu$ and γ gives that

$$\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[d, "e_f"'] & \bullet \arrow[d, "e_g"] \\
F(f) \arrow[r, "F(u{,}v)"] \arrow[d, "m_f"'] & F(g) \arrow[d, "m_g"] \\
\bullet \arrow[r, "v"] & \bullet
\end{tikzcd} \qquad (31)$$

commutes whenever (1) does. We put $\mathcal{E}_F := \{h \mid m_h \text{ is an isomorphism}\}$ and $\mathcal{M}_F := \{h \mid e_h \text{ is an isomorphism}\}$.

2.2. For the remainder of the paper, *we assume that every weak factorization system $F : \mathcal{K}^2 \to \mathcal{K}$ satisfies the identity $FE_{\mathcal{K}} = \mathrm{Id}_{\mathcal{K}}$ strictly, not just up to isomorphism.*

This is in fact no loss of generality since for every functor $F : \mathcal{K}^2 \to \mathcal{K}$ and every natural isomorphism $\gamma : \mathrm{Id}_{\mathcal{K}} \to FE_{\mathcal{K}}$ we can find a functor $F' : \mathcal{K}^2 \to \mathcal{K}$ with $F'E_{\mathcal{K}} = \mathrm{Id}_{\mathcal{K}}$, $F' \simeq F$ and $\mathcal{E}_{F'} = \mathcal{E}_F$, $\mathcal{M}_{F'} = \mathcal{M}_F$. One simply defines F' on objects by $F'(1_A) := A$ for every object A, and $F'(f) := F(f)$ if f is not an identity morphism, and for a morphism $(u, v) : f \to g$ one puts $F'(u, v) = \delta_g^{-1} F(u, v)\delta_f$ with $\delta_f := \gamma_A$ for $f = 1_A$ and $\delta_f = 1_{Ff}$ if f is not an identity morphism. It is easy to verify that F' is a functor and $\delta : F' \to F$ a natural isomorphism satisfying the properties claimed.

2.3. Let us now re-write 2.1 strictly in 2-categorical terms. A weak factorization system F (so that $FE = \mathrm{Id}_{\mathcal{K}}$) gives natural transformations

$$e := F\eta : \partial_0 \to F \quad \text{and} \quad m := F\mu : F \to \partial_1 \tag{32}$$

such that

$$eE = 1\,, \quad mE = 1\,, \quad \kappa = m \cdot e\,. \tag{33}$$

Hence F belongs to a factorization of the natural transformation κ.

Conversely, for arbitrary transformations e and m which satisfy (33), one has $e \cdot \partial_0\eta = F\eta \cdot eE\partial_0$ and $\partial_1\mu \cdot m = mE\partial_1 \cdot F\mu$; since $\partial_0\eta = 1$ and $\partial_1\mu = 1$ (see (16)), necessarily $e = F\eta$ and $m = F\mu$ follows.

2.4. With every weak factorization system F there are associated endofunctors

$$\begin{aligned} F_L &= F^2L : \mathcal{K}^2 \to \mathcal{K}^2\,, \qquad f \mapsto m_f\,,\\ F_R &= F^2R : \mathcal{K}^2 \to \mathcal{K}^2\,, \qquad f \mapsto e_f\,, \end{aligned} \tag{34}$$

and natural transformations

$$\begin{aligned} \lambda &: \mathrm{Id}_{\mathcal{K}^2} \to F_L\,, \qquad \lambda_f = (e_f, 1) : f \to m_f\,,\\ \rho &: F_R \to \mathrm{Id}_{\mathcal{K}^2}\,, \qquad \rho_f = (1, m_f) : e_f \to f\,, \end{aligned} \tag{35}$$

exhibiting F_L as a pointed endofunctor and F_R as a copointed endofunctor.

One has

$$F_LE = E\,, \qquad \lambda E = 1_E\,, \quad \partial_1 F_L = \partial_1\,, \quad \partial_1\lambda = 1_{\partial_1}\,, \tag{36}$$

$$F_RE = E\,, \qquad \rho E = 1_E\,, \quad \partial_0 F_R = \partial_0\,, \quad \partial_0\rho = 1_{\partial_0}\,, \tag{37}$$

Furthermore, F can be recovered from both F_L and F_R, since also

$$\partial_0 F_L = F = \partial_1 F_R\,. \tag{38}$$

2.5. For future reference we note that every weak factorization system F of $\mathcal{X}$ satisfies the identities

$$F^2(E_{\mathcal{X}})^2 = \mathrm{Id}_{\mathcal{X}^2}, \qquad F^2E_{\mathcal{X}^2} = E_{\mathcal{X}}F. \tag{39}$$

Furthermore, with φ and ψ of (17), one easily checks the identities

$$F^2\varphi(E_{\mathcal{X}})^2 = \lambda, \qquad F^2\varphi E_{\mathcal{X}^2} = \eta F_L, \tag{40}$$

$$F^2\psi(E_{\mathcal{X}})^2 = \rho, \qquad F^2\psi E_{\mathcal{X}^2} = \mu F_R, \tag{41}$$

2.6. By the first identity of (19), $M = M_{\mathcal{X}}$ *is a weak factorization system of* $\mathcal{X}^2$. It turns out that the data and identities of 2.3 and 2.4 will be of particular interest in case $F = M$. Evaluation of the transformations (32) at $(u, v) : f \to g$, considered as an object of $(\mathcal{X}^2)^2$, gives

$$e_{(u,v)} = (1, v) : f \to d \quad \text{and} \quad m_{(u,v)} = (u, 1) : d \to g \tag{42}$$

with $d = gu = vf$, see (3). Hence (34) defines the functor

$$M_R = (M_{\mathcal{X}})^2R_{\mathcal{X}^2} : (\mathcal{X}^2)^2 \to (\mathcal{X}^2)^2, \qquad (u, v) \mapsto e_{(u,v)}, \tag{43}$$

and (35) describes the natural transformation

$$\rho^M : M_R \to \mathrm{Id}_{(\mathcal{X}^2)^2}, \qquad \rho^M_{(u,v)} = (1, m_{(u,v)}) : e_{(u,v)} \to (u, v). \tag{44}$$

For a weak factorization system F of $\mathcal{X}$, the transformation $\rho^F = \rho$ of (35) is indeed determined by ρ^M, since

$$\rho^F = F^2\rho^M(E_{\mathcal{X}})^2. \tag{45}$$

In fact, domains and codomains of these transformations coincide by (28) and (19), and for every $\mathcal{X}$-morphism f one has with (21), (22) and (27)

$$e_{(E_{\mathcal{X}})^2f} = \eta_f \quad \text{and} \quad m_{(E_{\mathcal{X}})^2f} = \mu_f, \tag{46}$$

hence (44) gives

$$\rho^M_{(E_{\mathcal{X}})^2f} = (1, \mu_f) : \eta_f \to (E_{\mathcal{X}})^2f.$$

Application of F^2 finally gives with (32)

$$F^2\rho^M_{(E_{\mathcal{X}})^2f} = (1, m_f) : e_f \to f,$$

and this proves (45).

3. Eilenberg–Moore factorization systems

3.1. An *Eilenberg–Moore factorization system* (*E–M system*, for short) of a category $\mathscr{X}$ is a weak factorization system F (i.e., a functor $F : \mathscr{X}^2 \to \mathscr{X}$ with $FE_{\mathscr{X}} = \mathrm{Id}_{\mathscr{X}}$) such that, for every morphism f, one has $e_f \in \mathscr{E}_F$ and $m_f \in \mathscr{M}_F$ (cf. 2.1). The latter condition means that the natural transformations

$$mF_R : FF_R \to F \quad \text{and} \quad eF_L : F \to FF_L \tag{47}$$

are isomorphisms (cp. (34) and (38)).

An essential step for proving Theorem A is provided by the following lemma:

3.2. Lemma. *For an E–M system F, one has the identities*

$$mF_R = F\rho \quad \textit{and} \quad eF_L = F\lambda \,. \tag{48}$$

Proof. For every f, we must prove the identities

$$m_{e_f} = F\rho_f \quad \text{and} \quad e_{m_f} = F\lambda_f \,, \tag{49}$$

with $\rho_f = (1, m_f) : e_f \to f$ and $\lambda_f = (e_f, 1) : f \to m_f$ (see (35)).
The diagram

$$\begin{tikzcd}
\bullet \arrow[r, "e_f"] \arrow[d, "e_f"'] & \bullet \arrow[d, "e_{m_f}"] \\
\bullet \arrow[r, "s"] \arrow[d, "m_f"'] & \bullet \arrow[d, "m_{m_f}"] \\
\bullet \arrow[r, "1"] & \bullet
\end{tikzcd} \tag{50}$$

commutes for both $s = e_{m_f}$ and $s = F\lambda_f$. Hence the $\mathscr{X}^2$-morphism $(e_{m_f}e_f, m_f) : e_f \to m_{m_f}$ factors as

$$e_f \xrightarrow{(e_f, 1)} 1 \xrightarrow{(s, m_f)} m_{m_f} \,,$$

for each choice of s, with $(e_f, 1) = \mu_{e_f}$ (see (22)). Since $F\mu_{e_f} = m_{e_f}$, when applying F to the two decompositions, we obtain

$$F(e_{m_f}, m_f) \cdot m_{e_f} = F(F\lambda_f, m_f) \cdot m_{e_f} \,.$$

Since m_{e_f} is an isomorphism, we see that $\bar{s} := F(s, m_f)$ does not depend on the choice of s. But $\bar{s}$ makes the diagram

$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "1"'] & \bullet \arrow[d, "e_{m_{m_f}}"] \\ \bullet \arrow[r, "\bar{s}"] \arrow[d, "1"'] & \bullet \arrow[d, "m_{m_{m_f}}"] \\ \bullet \arrow[r, "m_f"] & \bullet \end{tikzcd} \tag{51}$$

commute, and $e_{m_{m_f}}$ is an isomorphism, hence s is independent of the two choices provided above as well. This proves the second of the identities (18); the first identity follows dually. □

We can now prove the non-trivial part of Theorem A.

3.3. Theorem. *For an E–M factorization system F of $\mathcal{X}$, the pair $(\mathscr{E}_F, \mathcal{M}_F)$ is an orthogonal factorization system of $\mathcal{X}$.*

Proof. We must show that for a commutative diagram (2), one necessarily has $t = F(u, v)$, with $(u, v) : f \to g$ in $\mathcal{X}^2$. First we note that the commutativity of (2) gives the following commutative diagram in $\mathcal{X}^2$:

$$\begin{tikzcd} & f \arrow[r, "(u,v)"] & g \arrow[dr, "\lambda_g"] & \\ e_f \arrow[ur, "\rho_f"] \arrow[dr, "\mu_{e_f}"'] & & & m_g \\ & 1 \arrow[r, "(t,t)"] & 1 \arrow[ur, "\eta_{m_g}"'] & \end{tikzcd} \tag{52}$$

Application of F to (52) gives, with (48),

$$\begin{aligned} e_{m_g} \cdot F(u, v) \cdot m_{e_f} &= F\lambda_g \cdot F(u, v) \cdot F\rho_f \\ &= F\eta_{m_g} \cdot F(t, t) \cdot F\mu_{e_f} \\ &= e_{m_g} \cdot t \cdot m_{e_f} . \end{aligned}$$

Since e_{m_g} and m_{e_f} are isomorphisms, $t = F(u, v)$ follows. □

3.4. Corollary. *For an E–M factorization system F of $\mathcal{X}$, both classes $\mathscr{E}_F$ and $\mathcal{M}_F$ are closed under composition, $\mathscr{E}_F \cap \mathcal{M}_F$ is the class of isomorphisms of $\mathcal{X}$, $\mathscr{E}_F$ is closed under colimits in $\mathcal{X}$, and $\mathcal{M}_F$ is closed under limits in $\mathcal{X}$.* □

3.5. As Im and Kelly [4] showed, stability under colimits and limits yields in particular the cancellation rules

$$\text{if } f \cdot g \in \mathscr{E}_F \text{ and } g \in \mathscr{E}_F , \quad \text{then } f \in \mathscr{E}_F ; \tag{53}$$

$$\text{if } f \cdot g \in \mathscr{M}_F \text{ and } f \in \mathscr{M}_F , \quad \text{then } g \in \mathscr{M}_F . \tag{54}$$

Only these rules and closure under composition are needed to derive from the commutative diagram (2) the following rules that will be used in Section 4.

Corollary. *For an E–M factorization system F and every $\mathscr{K}^2$-morphism (u, v) : $f \to g$, one has the following implications*:

$$\textit{if } u \in \mathscr{E}_F , \quad \textit{then } F(u, v) \in \mathscr{E}_F ; \tag{55}$$

$$\textit{if } v \in \mathscr{M}_F , \quad \textit{then } F(u, v) \in \mathscr{M}_F . \quad \square \tag{56}$$

3.6. In [2], a weak factorization system satisfying the uniqueness condition (III) of the Introduction, is called a (not necessarily orthogonal!) *factorization system* of $\mathscr{K}$. It is shown that, even in the absence of properties (I) or (II) of the Introduction, for a factorization system F one already obtains closure of $\mathscr{E}_F$ under colimits and of $\mathscr{M}_F$ under limits (but not in general closure under composition!).

An easy inspection of the proofs of Lemma 3.2 and Theorem 3.3 reveals that they work already under the hypothesis that

$$m_{e_f} \text{ is epic and } e_{m_f} \text{ is monic} , \tag{57}$$

for every morphism f, rather than under the assumption that these morphisms are iso. In other words, in Theorem 3.3 we showed that a weak factorization system F which satisfies (57) is a factorization system in the sense of [2]. This leads to a strengthening of the last part of Corollary 3.4:

Corollary. *For a weak factorization system F which satisfies property* (57) *for every morphism f, the class $\mathscr{E}_F$ is closed under colimits and $\mathscr{M}_F$ is closed under limits.* $\square$

4. E–M factorization systems as Eilenberg–Moore algebras

4.1. A (one-dimensional) algebra-structure on $\mathscr{K}$ with respect to the monad (Φ, E, M) (see 1.4) would be a weak factorization system $F : \mathscr{K}^2 \to \mathscr{K}$ which satisfies the additional identity $FM_{\mathscr{K}} = FF^2$. Clearly, in the 2-category **Cat**, this identity should be relaxed to an isomorphism which may have to satisfy some identities itself. In order to clarify the situation, we first discuss what it means for F to admit just a natural transformation $\alpha : FM_{\mathscr{K}} \to FF^2$.

4.2. Lemma. *For a weak factorization system F on $\mathscr{K}$ and any natural transformation $\alpha : FM \to FF^2$ one has*

$$F\lambda \cdot \alpha(E_{\mathcal{H}})^2 = \alpha L = eF_L \cdot \alpha E_{\mathcal{H}^2}\,, \tag{58}$$

$$F\rho \cdot \alpha R = \alpha(E_{\mathcal{H}})^2 \quad \textit{and} \quad mF_R = \alpha E_{\mathcal{H}^2} \cdot F\rho\,. \tag{59}$$

Proof. Naturality of the transformations α and φ (of (17)) gives

$$\alpha L \cdot FM\varphi(E_{\mathcal{H}})^2 = FF^2\varphi(E_{\mathcal{H}})^2 \cdot \alpha(E_{\mathcal{H}})^2\,,$$

$$\alpha L \cdot FM\varphi E_{\mathcal{H}^2} = FF^2\varphi E_{\mathcal{H}^2} \cdot \alpha E_{\mathcal{H}^2}\,.$$

Since $M\varphi = 1$ (see (18)), with the identities (32) and (40), one obtains (58). Similarly, using the transformation ψ and the identities (41), one derives (59). □

4.3. An *Eilenberg–Moore algebra F of $\mathcal{H}$* is a weak factorization system F of $\mathcal{H}$ such that there is a natural isomorphism $\alpha : FM_{\mathcal{H}} \xrightarrow{\sim} FF^2$ with

$$\alpha E_{\mathcal{H}^2} = 1_F = \alpha(E_{\mathcal{H}})^2 \tag{60}$$

(domain and codomain of the transformations in (60) coincide by (19) and (39)).

More precisely, these are the *normal pseudo-algebras* with respect to the monad (13) on **Cat**; here 'pseudo' allows the associativity diagram to commute only up to coherent isomorphism, while 'normal' means that the unit axiom holds strictly.

Hence, for an Eilenberg–Moore algebra one obtains from (58), (59) and (60) that

$$eF_L = F\lambda = \alpha L \quad \text{and} \quad mF_R = F\rho = (\alpha R)^{-1} \tag{61}$$

are isomorphisms. This means in particular that F is an E–M factorization system (see (47)), and it proves the first half of the main theorem of the paper:

4.4. Theorem. *The E–M factorization systems of* 3.1 *are exactly the Eilenberg–Moore algebras of* 4.3.

Proof. For an E–M factorization system F we must construct an appropriate transformation α.

For the functors F_R of (34) and M_R of (43), from (28) we obtain

$$F^2M_R = F_RM_{\mathcal{H}}\,, \tag{62}$$

and we can define α as the composite

$$FM_{\mathcal{H}} = \partial_1 F_R M_{\mathcal{H}} \xrightarrow{(mF_RM_{\mathcal{H}})^{-1}} FF_RM_{\mathcal{H}} = FF^2M_R \xrightarrow{FF^2\rho M} FF^2\,; \tag{63}$$

here we use the identity $F = \partial_1 F_R$ (see (38)) and the fact that mF_R is invertible (see (47)). The description (44) of the transformation ρ^M is used to conclude from (55) and (56) that $FF^2\rho^M$ belongs pointwise to $\mathscr{E}_F \cap \mathscr{M}_F$ and is therefore an isomorphism. Hence α is a well-defined isomorphism.

We must verify the identities (60). First, the identities $F_R E = E$ and $\rho E = 1_E$ of (37), in the case of the E–M factorization system $M_{\mathscr{K}}$, show

$$M_R E_{\mathscr{K}^2} = E_{\mathscr{K}^2} \quad \text{and} \quad \rho^M E_{\mathscr{K}^2} = 1 . \tag{64}$$

Hence one has, with (62), (39) and (33),

$$mF_R M_{\mathscr{K}} E_{\mathscr{K}^2} = mF^2 M_R E_{\mathscr{K}^2} = mF^2 E_{\mathscr{K}^2} = mE_{\mathscr{K}} F = 1 ,$$

and then

$$\alpha E_{\mathscr{K}^2} = FF^2 \rho^M E_{\mathscr{K}^2} \cdot (mF_R M_{\mathscr{K}} E_{\mathscr{K}^2})^{-1} = 1 \cdot 1 = 1 .$$

The identities (19), (45) and (48) give

$$\begin{aligned} \alpha(E_{\mathscr{K}})^2 &= FF^2\rho^M(E_{\mathscr{K}})^2 \cdot (mF_R M_{\mathscr{K}}(E_{\mathscr{K}})^2)^{-1} \\ &= F\rho^F \cdot (F\rho^F)^{-1} = 1 . \end{aligned}$$

This completes the proof. □

4.5. We remark that the proof of Theorem 4.4 uses almost entirely 2-categorical methods, with the exception of the proofs of (48), (55) and (56) which are being used.

Notes added in proof

A. Blass remarked that the proof of Theorem 3.3 can be simplified. One considers just the lower part of diagram (52) and observes that one can trade (t, t) for $(F(u, v), F(u, v))$. Lemma 3.2 then becomes a corollary of Theorem 3.3.

Any natural isomorphism $\alpha : FM \to FF^2$ must satisfy the identities (60). Indeed, from (58) and (59) one obtains that mF_R and eF_L are iso, so that F must be an E–M factorization system. Hence one can use (50) to derive (60) from (58) and (59). This makes the second part of the proof of Theorem 4.4 redundant.

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